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ASYMPTOTIC PROPERTIES OF BETTI NUMBERS OF MODULES OVER CERTAIN RINGS

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Introduction

Throughout this paper, all rings are assumed to be commutative noetherian local rings with common residue field k.

Let $(R,\mathfrak{m})$ be a local ring with maximal ideal $\mathfrak{m}$. If M is a finitely generated R-module, the Betti numbers of M are the integers $b_i(M)=\dim_k \operatorname{Tor}_i^R(M,k)$. The purpose of this paper is to study the asymptotic behaviour of the sequence $b_i(M)$ over two classes of rings: Rings of the form $S/J\mathfrak{n}$, $(S,\mathfrak{n})$ local ring and J ideal of S, and rings $(R,\mathfrak{m})$ with $\mathfrak{m}^3=0$.

In [1, 5.8] Avramov states the following problem:

(1) Is the sequence $b_i(M)$ eventually non-decreasing for any finitely generated module M over the local ring R?

In [9] Ramras considers a more limited question:

(2) Is it true that for an arbitrary finitely generated module over a local ring R, there are only two possibilities: either the sequence $b_i(M)$ is eventually constant, or $\lim_i b_i(M)=\infty$?

We give a positive answer to (2) for the classes of rings we study and a positive answer to (1) for the local rings with $\mathfrak{m}^3=0$.

We are also interested in the rate of growth of the sequences of Betti numbers, and the following definition will be useful:

Definition. We say that the sequence $b_i(M)$ has *exponential growth* if there exist an integer n_0 and real numbers $C>A>1$ so that for each $i\geq n_0$ the double inequality:

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$$A^i \leq b_i(M) \leq C^i$$

is satisfied.

If for every non-free finitely generated R-module M the sequence $b_i(M)$ has exponential growth and if n_0 and A can be chosen independently of M, we say that the sequences of Betti numbers have *uniform exponential growth* over R.

We also give information about the radius of convergence r_M of the Poincaré series $P_R^M(t) = \sum_{i\geq 0} b_i(M)t^i$. We always have the lower bound $r_M \geq r_k$, and, of course, exponential growth gives an upper bound.

Our main results are the following:

A. *Rings of the form* $R = S/J\mathfrak{n}$

Let $(S, \mathfrak{n})$ be a local ring, J be an ideal and $R = S/\mathfrak{n}J$. Such rings have already been studied by Ramras [8] and Gover and Ramras [4]. In [4] they proved that the sequence $(b_i(M))_{i\geq 1}$ is non-decreasing provided that J is a nonnilpotent ideal; consequently, question (2) has a positive answer in this case. We prove here more generally:

Theorem A.3. *Let $R = S/\mathfrak{n}J$ and assume that $\mathfrak{n}J \neq 0$. Then, for any finitely generated R-module M, either* $\lim_i b_i(M) = \infty$ *or $b_i(M)$ is eventually constant. Moreover the sequences $(b_{2i}(M))_{i\geq 1}$ and $(b_{2i+1}(M))_{i\geq 0}$ are non-decreasing.*

With the hypothesis $\dim_k \mathfrak{n}J/\mathfrak{n}^2J \geq 2$, we have more information on the Betti numbers:

Theorem A.2. *Let $R = S/\mathfrak{n}J$ and suppose* $\dim_k \mathfrak{n}J/\mathfrak{n}^2J \geq 2$. *Then, the sequences of Betti numbers have uniform exponential growth over R. For all finitely generated non-free R-modules M, we have $r_k \leq r_M \leq \sqrt{2}/2$, and the sequences $(b_{2i}(M))_{i\geq 1}$ and $(b_{2i+1}(M))_{i\geq 0}$ are strictly increasing.*

In [8] Ramras has produced examples of rings over which the Betti numbers are strictly increasing, from b_1 onwards. We prove the new result:

Theorem A.1. *Let $(S, \mathfrak{n})$ be a local ring of Krull dimension $d \geq 2$. Let J be a $\mathfrak{n}$-primary ideal and let $R = S/\mathfrak{n}J$. Then, for all finitely generated non-free R-modules M, the sequence $(b_i(M))_{i\geq 1}$ is strictly increasing.*

This theorem applies particularly to the case $R = S/\mathfrak{n}^p$, $p \geq 2$ and $\dim S \geq 2$ (This generalizes one of Gover and Ramras's results [4, Theorem 1.1]).

B. *Rings* $(R, \mathfrak{m})$ *with* $\mathfrak{m}^3 = 0$

We prove that most of these rings have the following interesting property:

If M is a finitely generated non-free R-module, then k is a direct summand of one of the modules of syzygies of M.

From this property and after examining some particular cases, we obtain:

Theorem B. *Let* $(R, \mathfrak{m})$ *be a local ring with* $\mathfrak{m}^3 = 0$. *Let* $n = \dim_k \mathfrak{m}/\mathfrak{m}^2$ *and* $a = \dim_k \mathfrak{m}^2$. *Assuming that the socle of R is* $\mathfrak{m}^2$, *then:*

(1) *Suppose* $n \neq 1$ *and* $(n, a) \neq (2, 1)$. *If* $a \neq n-1$, *then the sequences of Betti numbers have exponential growth. If* $a > n$ *or if* $P_R^k(t) \neq (1 - nt + at^2)^{-1}$, *then these sequences have uniform exponential growth. If* $a = n - 1$ *and* $P_R^k(t) = (1 - nt + at^2)^{-1}$, *then for any finitely generated non-free R-module M, the sequence* $(b_i(M))_{i \geq 1}$ *is either stationary or has exponential growth.*

(2) *Let M be any finitely generated non-free R-module. If* $P_R^k(t) \neq (1 - nt + at^2)^{-1}$, *then* $r_M = r_k$. *If* $P_R^k(t) = (1 - nt + at^2)^{-1}$, *then either* $r_M = r_k$ *or else* $a \geq n - 1$, $P_R^k(t) = (1 - r_1 t)^{-1}(1 - r_2 t)^{-1}$, r_1 *and* r_2 *are integers, and* $r_M = r_1^{-1} > r_2^{-1} = r_k$.

(3) *If the sequence* $(b_i(M))_{i \geq 1}$ *is not stationary, then there exists an integer j so that the sequence* $(b_i(M))_{i \geq j}$ *is strictly increasing* ($j = 1$ *if* $a > n$).

With the hypothesis of Theorem B we have the following corollary:

Corollary. *Suppose that the radius of convergence* r_k *of* $P_R^k(t)$ *is transcendental. Then, if M is a finitely generated non-free R-module, the Poincaré séries* $P_R^M(t)$ *is not rational.*

The following proposition complements Theorem B:

Proposition 3.9. *Let* $(R, \mathfrak{m})$ *be a local ring with* $\mathfrak{m}^3 = 0$, *and* $\mathfrak{m}^2 \neq 0$. *Assume* $\mathrm{soc}(R) \neq \mathfrak{m}^2$. *Then:*

(1) *The sequences of Betti numbers have uniform exponential growth.*

(2) *For all non-free R-modules M, we have* $r_M = r_k$ *and the sequence* $(b_i(M))_{i \geq 1}$ *is strictly increasing.*

The proofs of the theorems are given in Sections 2 and 3.

1. Some preliminaries

Here, and in all that follows, the modules are assumed to be finitely generated. For an R-module M let $P_\cdot : \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0$ be a minimal free resolution of M. Thus, $b_i(M)$ is also the rank of the free module P_i. The modules of syzygies of M are defined for $n \geq 1$ by: $\mathrm{syz}^n(M) = \mathrm{Im}(P_n \to P_{n-1})$. By minimality, we have $\mathrm{syz}^n(M) \subset \mathfrak{m} P_{n-1}$. It is well-known that $b_i(\mathrm{syz}^j(M)) = b_{i+j}(M)$.

We now observe that there exists a lower bound for the radius of convergence of the Poincaré series, and that the Betti numbers are exponentially bounded from above.

Proposition 1.1. *Let* $(R, \mathfrak{m})$ *be a local ring, and* M *be a* R*-module. Then:*

(1) *The radius of convergence* r_M *of* $P_R^M(t)$ *satisfies the inequality* $r_M \geq r_k$.

(2) *There exists a constant* $C > 1$ *so that for* $i \geq 1$, $b_i(M) \leq C^i$.

First we need a lemma:

Lemma. *Let* M *be an* R*-module. Then, there exists a constant* D *so that for* $i \geq 0$, $b_i(M) \leq D b_i(k)$.

Proof. For a module of finite length, we can choose $D = l(M)$ ($l(M)$ denotes the length of a module). This follows easily by induction on $l(M)$. For an arbitrary M, by a result of levin [7], there exists for a sufficiently large integer p, an injective map $\mathrm{Tor}_*^R(M,k) \to \mathrm{Tor}_*^R(M/\mathfrak{m}^p M, k)$, induced by the projection $M \to M/\mathfrak{m}^p M$. This yields

$$b_i(M) \leq b_i(M/\mathfrak{m}^p M) \leq l(M/\mathfrak{m}^p M) b_i(k).$$

We can now prove the proposition.

(1) is obvious from the lemma. For $M = k$, (2) is well-known (and results from the classical inequality for $P_R^k(t)$ due to Serre). Now, for an arbitrary M, (2) follows easily from the lemma.

2. Rings of the form $S/\mathfrak{n}J$

Let $(S, \mathfrak{n})$ be a local ring and J be an ideal of S.

Let $R = S/\mathfrak{n}J$, and $\mathfrak{m}$ its maximal ideal. Since every syzygy module over R is a submodule of some $\mathfrak{m}R^b = \mathfrak{n}R^b$, every syzygy module is annihilated by J (as S-module).

We shall prove Theorem A.1, first proving the following proposition:

Proposition 2.1. *Let* $(S, \mathfrak{n})$ *be a local ring and* J *be a* $\mathfrak{n}$*-primary ideal. Let* $c_{-1} = l(S/J)$, $c_0 = l(J/\mathfrak{n}J)$ *and for* $i \geq 1$, $c_i = l(\mathfrak{n}^i J/\mathfrak{n}^{i+1} J)$. *Assume that for some integer* $i \geq 0$ *we have* $c_i > c_{-1}$. *Then, for every non-free module* M *over* $R = S/\mathfrak{n}J$, *the sequence* $(b_i(M))_{i \geq 1}$ *is strictly increasing, and the sequences of Betti numbers have uniform exponential growth over* R.

Proof. Let $\tilde{R} = S/\mathfrak{n}^{i+1}J$, and $\tilde{\mathfrak{m}}$ its maximal ideal. Let N be a non-free R-module and $M = \mathrm{syz}^p(N)$. With K and L denoting the first syzygy modules of M respectively over R and $\tilde{R}$, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & \tilde{R}^b & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \wr} & & \\ 0 & \longrightarrow & K & \longrightarrow & R^b & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where $b=b_0(M)$, $L\subset\tilde{\mathfrak{m}}\tilde{R}^b$, $K\subset\mathfrak{m}R^b$. As Ramras remarked [8, Theorem 3.2], K and L have the same minimal number of generators $b_1=b_1(M)$. In order to show this, we observe that, since M is annihilated by J, L contains $J\tilde{R}^b$. We thus have $\ker f=\mathfrak{n}(J\tilde{R}^b)\subset\mathfrak{n}L$. Consequently, the surjective map f induces an isomorphism:

$$L/\tilde{\mathfrak{m}}L=L/\mathfrak{n}L\xrightarrow{\sim}K/\mathfrak{n}K=K/\mathfrak{m}K=k^{b_1}.$$

We then look for bounds for the length of L. Since L is a syzygy module over $\tilde{R}$, it is annihilated by $\mathfrak{n}^iJ$ and thus is a quotient of $(S/\mathfrak{n}^iJ)^{b_1}$ (we use the convention $\mathfrak{n}^0=S$). This yields the inequality:

$$l(L)\leq b_1 l(S/\mathfrak{n}^iJ)=b_1(c_{-1}+c_0+\cdots+c_{i-1}).$$

On the other hand, L contains $J\tilde{R}^b\simeq(J/\mathfrak{n}^{i+1}J)^b$ and consequently:

$$l(L)\geq bl(J/\mathfrak{n}^{i+1}J)=b(c_0+\cdots+c_i).$$

We can now write for $p\geq 1$:

$$b_{p+1}(N)/b_p(N)=b_1(M)/b_0(M)\geq(c_0+\cdots+c_i)/(c_{-1}+\cdots+c_{i-1})=e>1$$

and the desired conclusion follows.

We can now prove Theorem A.1.

2.2. Proof of Theorem A.1. Since the ideal J is $\mathfrak{n}$-primary, its Krull dimension as an S-module is equal to the Krull dimension of S, and so, $\dim_K J=d\geq 2$. It is well-known that for large i, $c_i=l(\mathfrak{n}^iJ/\mathfrak{n}^{i+1}J)$ is a polynomial in i of degree $d-1$. It follows that $\lim_i c_i=\infty$. In particular, there exists i such that $c_{-1}<c_i$, and the preceding proposition applies.

We now give the proof of Theorem A.2.

2.3. Proof of Theorem A.2. Let N be an R-module and $M=\operatorname{syz}^p(N)$. Let $\tilde{R}=S/\mathfrak{n}^2J$. As in Proposition 2.1, K and L denote the first syzygy modules of M over R and $\tilde{R}$. Since L is annihilated by $\mathfrak{n}J$, L is also an R-module. We now have an exact sequence of R-modules:

$$0\to(\mathfrak{n}J/\mathfrak{n}^2J)^b\to L\xrightarrow{f}K\to 0$$

where $b=b_0(M)$. Since K and L have the same minimal number of generators, this sequence yields a surjective map: $\mathrm{Tor}_1^R(K,k)\to\mathrm{Tor}_0^R((\mathfrak{n}J/\mathfrak{n}^2J)^b,k)$. Letting $c=\dim_k(\mathfrak{n}J/\mathfrak{n}^2J)$, we obtain:

$$b_{p+2}(N)=b_1(K)\geq cb_0(M)=cb_p(N).$$

Since we suppose $c\geq 2$, it follows that the sequences $(b_{2i}(N))_{i\geq 1}$ and $(b_{2i+1}(N))_{i\geq 0}$ are strictly increasing. We also have the inequality: $r_M\leq\sqrt{2}/2$. For $i\geq 3$ we can also write: $b_i(N)\geq A^i$ with $A=c^{1/4}>1$. The proof of the theorem is thus complete.

We conclude this section with the proof of Theorem A.3.

2.4. Proof of Theorem A.3. In view of Theorem A.2 it is enough to consider the case $\dim_k(\mathfrak{n}J/\mathfrak{n}^2J)=1$. Let N be an R-module. Using the preceding proof we see that the sequences $(b_{2i}(N))_{i\geq 1}$ and $(b_{2i+1}(N))_{i\geq 0}$ are non-decreasing. We must prove that if one of these sequences is eventually constant, then so is the sequence $b_i(N)$. Since $b_i(N)=\mathrm{rank}\,P_i$, where P_i is the i-th free module in a minimal free resolution of N, it suffices to establish the following lemma.

Lemma. *Let R be a local ring, and let*

$$F_\bullet:\ \cdots\to F_{2i+1}\to F_{2i}\to F_{2i-1}\to\cdots\to F_0\to 0$$

be a complex of finitely generated free R-modules, which is exact except (possibly) at F_0. Suppose that for all i we have $\mathrm{rank}\,F_{2i}=\mathrm{rank}\,F_0=a$ and $\mathrm{rank}\,F_{2i+3}\geq\mathrm{rank}\,F_{2i+1}$. Then the sequence $(\mathrm{rank}\,F_i)_{i\geq 0}$ is eventually constant.

Proof. As in [2, Proposition 5.3], by localizing at a minimal prime of R (this does not affect the rank of F_i) we may assume R to be artinian, and thus of finite length. Now

$$l(F_{2i+1})=\mathrm{rank}\,F_{2i+1}\cdot l(R)\leq l(F_{2i+2})+l(F_{2i})=2a\,l(R).$$

Consequently the sequence $(\mathrm{rank}\,F_{2i+1})_{i\geq 0}$ is bounded above and thus, is eventually constant. By removing a right-hand part from the complex $F_\bullet$, we may assume that for all i, $\mathrm{rank}\,F_{2i+1}=b$. Letting $A_i=\mathrm{coker}(F_{i+1}\to F_i)$ the exact sequence:

$$0\to A_{2n+1}\to F_{2n}\to F_{2n-1}\to\cdots\to F_0\to A_0\to 0$$

yields:

$$\sum_{i=0}^{i=n} l(F_{2i})=\sum_{i=1}^{i=n} l(F_{2i-1})+l(A_{2n+1})+l(A_0),$$

or,

$$(n+1)a\,l(R)=nb\,l(R)+l(A_{2n+1})+l(A_0).$$

Since $l(A_{2n+1})\leq a\,l(R)$, we have $a-b\leq[l(A_0)/l(R)]/n$. As n can be arbitrarily large, we have $a\leq b$. But this same argument may be applied to the sequence

$$0\to A_{2n+2}\to F_{2n+1}\to F_{2n}\to\cdots\to F_1\to A_1\to 0$$

in which the roles of a and b are reversed, so that we get $b \leq a$. Hence $a = b$.

3. Rings $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$

Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m}^3 = 0$. We suppose that $\mathrm{soc}(R) = \mathfrak{m}^2$ (here and below, $\mathrm{soc}(\cdot) = \mathrm{socle}(\cdot)$).

Let $n = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ and $a = \dim_k(\mathfrak{m}^2)$. R is a complete intersection if, and only if, $n = 1$ or $(n, a) = (2, 1)$. Gulliksen has shown [5] that over a complete intersection the sequences of Betti numbers are polynomially bounded from above (and thus, they do not have exponential growth). For this reason we assume that $n \neq 1$ and $(n, a) \neq (2, 1)$ in the first part of Theorem B. It should also be noted that, if $n \neq 1$, the sequence $(b_i(k))_{i \geq 0}$ is strictly increasing.

If M is a *syzygy* module over R, then $\mathfrak{m}^2 M = 0$, and we shall prove some lemmas for modules satisfying this last condition.

First, we need a definition

Definition 3.1. Let M be a non-zero R-module so that $\mathfrak{m}^2 M = 0$. We say that M is p-exceptional, $p \geq 1$, if the R-module k is not a direct summand of the modules $\mathrm{syz}^1 M, \ldots, \mathrm{syz}^p M$. If M is p-exceptional for every p, we say that M is exceptional. If k is exceptional, we say that the ring R is exceptional.

Lemma 3.3 and Lemma 3.6, will justify this terminology.

We begin with an obvious lemma, available for every local ring.

Lemma 3.2. *Let M be a R-module. The following statements are equivalent:*

(1) *The R-module k is not a direct summand of M.*

(2) *We have the inclusion* $\mathrm{soc}(M) \subset \mathfrak{m} M$.

Notations. Let $f(t) = \sum_{n \geq 0} a_n t^n \in \mathbb{Z}[[t]]$ be a formal series; we write $f(t)|_p$ for the polynomial $\sum_{n=0}^{n=p} a_n t^n$. For a module M with $\mathfrak{m}^2 M = 0$, we let $s(M) = \dim_k(\mathfrak{m} M)$, thus, $l(M) = b_0(M) + s(M)$.

The following lemma gives a criterion for a module to be p-exceptional.

Lemma 3.3. *Let M be a non-zero R-module so that* $\mathfrak{m}^2 M = 0$. *Then, M is p-exceptional if and only if we have:*

$$P_R^M(t)|_p = ((b_0(M) - s(M)t)/(1 - nt + at^2))|_p.$$

Proof. Let $0 \to K \to R^b \to M \to 0$ be exact with $K \subset \mathfrak{m} R^b$ and $b = b_0(M)$. Since $\mathfrak{m}^2 M = 0$ we have $\mathrm{soc}(K) = \mathrm{soc}(R^b) = \mathfrak{m}^2 R^b$. Since $\mathfrak{m} K \subset \mathrm{soc}(K)$, by virtue of Lemma 3.2 we get:

k is not a direct summand of K if and only if $\mathfrak{m}K = \mathrm{soc}(K)$, or, since $b_0(K) = l(K) - s(K)$, if and only if $b_0(K) = l(K) - \dim_k \mathrm{soc}(K)$. But now,

$$b_0(K) = b_1(M), \qquad \dim_k \mathrm{soc}(K) = ab_0(M),$$

$$l(K) = l(R^b) - l(M) = nb_0(M) + ab_0(M) - s(M).$$

Thus, we can state: k is not a direct summand of $K = \mathrm{syz}^1(M)$ if and only if:

$$\mathrm{E}(M): \quad b_1(M) = nb_0(M) - s(M).$$

Now, M is p-exceptional if and only if the equalities $\mathrm{E}(M), \mathrm{E}(\mathrm{syz}^1(M)), \ldots, \mathrm{E}(\mathrm{syz}^{p-1}(M))$ hold. Since $b_0(\mathrm{syz}^i(M)) = b_i(M)$, $b_1(\mathrm{syz}^i(M)) = b_{i+1}(M)$, and noting that if k is not a direct summand of $\mathrm{syz}^i(M)$ then $s(\mathrm{syz}^i(M)) = ab_{i-1}(M)$, we can conclude that: M is p-exceptional if and only if:

$$b_1(M) = nb_0(M) - s(M),$$

$$b_i(M) = nb_{i-1}(M) - ab_{i-2}(M), \quad 2 \le i \le p.$$

But it is just another way to restate the lemma.

Corollary 3.4. (1) *A non-zero R-module M so that* $\mathfrak{m}^2M = 0$ *is exceptional if and only if* $P_R^M(t) = (b_0(M) - s(M)t)/(1 - nt + at^2)^{-1}$.

(2) *The ring R is exceptional if and only if* $P_R^k(t) = (1 - nt + at^2)^{-1}$.

Lemma 3.5. *Let M be a non-zero R-module such that* $\mathfrak{m}^2M = 0$. *Then, for* $j \ge 2$ *we have the inequality:* $b_j(M) \ge nb_{j-1}(M) - ab_{j-2}(M)$.

Proof. Let $N = \mathrm{syz}^{j-1}(M)$. From the exact sequence: $0 \to \mathfrak{m}N \to N \to N/\mathfrak{m}N \to 0$, we get: $b_1(N) \ge b_1(N/\mathfrak{m}N) - b_0(\mathfrak{m}N)$. Since $b_1(N) = b_j(M)$, $b_1(N/\mathfrak{m}N) = nb_{j-1}(M)$ and since $b_0(\mathfrak{m}N) \le ab_{j-2}(M)$, we have: $b_j(M) \ge nb_{j-1}(M) - ab_{j-2}(M)$.

We now come to a key lemma:

Lemma 3.6. *Let M be a non-zero R-module such that* $\mathfrak{m}^2M = 0$. *If M is p-exceptional, then k is p-exceptional. In particular, if M is exceptional, then so is the ring R.*

Proof. Since k is not a direct summand of $\mathfrak{m}$, k is always 1-exceptional. Now, assuming that k is $p-1$ exceptional and that there exists a p-exceptional module M, the exact sequence $0 \to \mathfrak{m}M \to M \to M/\mathfrak{m}M \to 0$ yields for all $j \ge 1$: $b_0(M)b_j(k) \le b_j(M) + s(M)b_{j-1}(k)$. By using Lemma 3.3, we can deduce from this, the following coefficientwise inequality: $P_R^k(t)|_p \le (1 - nt + at^2)^{-1}|_p$. In particular, we have $b_p(k) \le nb_{p-1}(k) - ab_{p-2}(k)$, and, by Lemma 3.5, this inequality is in fact an equality. It follows that k is p-exceptional.

Remark. From the proof above, it is easy to deduce that the induced map: $\mathrm{Tor}_j^R(\mathfrak{m}M,k)\to\mathrm{Tor}_j^R(M,k)$ is zero for $0\le j\le p$. For another proof of this, see [10, Lemma 4].

We are now in a position to prove Theorem B.

3.7. Proof of Theorem B. We begin by the proof of the assertions (1) and (2). Let M be a non-free R-module. Replacing, if necessary, M by $\mathrm{syz}^1(M)$, we may assume that $\mathfrak{m}^2M=0$.

(a) We first suppose that the ring R is not exceptional. Let p_0 be the smallest integer p such that $P_R^k(t)|_p\ne(1-nt+at^2)^{-1}|_p$. By virtue of Lemmas 3.3 and 3.6, there exists an integer j, $1\le j\le p_0$, such that k is a direct summand of $\mathrm{syz}^j(M)$. We get: $b_{l+j}(M)=b_l(\mathrm{syz}^j(M))\ge b_l(k)$. Since the sequence $b_i(k)$ is increasing, we also have:

$$b_{l+p_0}(M)\ge b_l(k).$$

Hence, we get $r_M\le r_k$, and by Proposition 1.1, $r_M=r_k$. Now, to obtain the uniform exponential growth, it is clearly enough to prove that:

There exist a real $C>1$ and an integer l such that for $i\ge l$

$$b_i(k)\ge C^i.$$

But, it has been proved by Avramov [1, Theorem 6.2] that this last statement is in fact a characteristic property of rings which are not complete intersections, or regular; and this applies to our particular case. (Here, C and l could also be obtained by elementary manipulations.)

(b) Let us now suppose that the ring R is exceptional, therefore, $P_R^k(t)=(1-nt+at^2)^{-1}$. If M is not exceptional, then as above, k is a direct summand of some $\mathrm{syz}^j(M)$ and we conclude that $r_M=r_k$ and that the sequence $b_i(M)$ has exponential growth.

Now, supposing M is exceptional, therefore, by Corollary 3.4,

$$P_R^M(t)=(b_0(M)-s(M)t)/(1-nt+at^2)^{-1}.$$

Thus, the sequence $u_i(M)=b_{i+1}(M)/b_i(M)$ is given from $u_0(M)$ by induction:

$$u_{i+1}(M)=f(u_i(M)),\qquad f(X)=n-a/X.$$

Consider, for $X>0$, the graphs of the curve $Y=f(X)=n-a/X$ and the line $Y=X$. Their intersections, if any, are precisely the fixed points of f, or equivalently, positive roots of the equation $X^2-nX+a=0$. Suppose this equation has no positive real root. Then the curve lies entirely below the line, i.e. for all $X>0$, $f(X)<X$. Since $u_i(M)>0$, the sequence $\{f^i(u_0(M))\}$ is both bounded below and decreasing, and therefore has a limit. This limit is necessarily a fixed point of f and so we have a contradiction. Thus the quadratic does have two positive real roots, say r_1 and r_2 with $r_1\le r_2$. Furthermore, as the smaller of the two roots of

$1-nX+aX^2=0$, r_2^{-1} must be the radius of convergence of $(1-nt+at^2)^{-1}=P_R^k(t)$ and hence $r_2=r_k^{-1}$.

Similar reasoning about f shows that $u_0(M)\geq r_1$. Then, either $u_0(M)=r_1$ and for every i, $u_i(M)=r_1$, or else r_2 is the limit of the sequence $u_i(M)$.

In the first case, r_1 and r_2 are necessarily positive integers, and consequently $a\geq n-1$. The sequence $b_i(M)$ is constant if $r_1=1$ (and then, $a=n-1$) and has exponential growth if $r_1>1$. We have $P_R^M(t)=b_0(M)\,(1-r_1t)^{-1}$ and $r_M=r_1^{-1}\geq r_2^{-1}=r_k$.

In the second case, $r_M=r_2^{-1}=r_k$. If $(n,a)\neq(2,1)$, $r_k>1$ and there exist an integer p and a real $C>1$ such that for $i>p$, $u_i(M)=b_{i+1}(M)/b_i(M)>C>1$. This shows that the sequence $b_i(M)$ has exponential growth.

So, we have established the assertions (1) and (2), except for the uniform exponential growth when R is exceptional and $a>n$. This will be proved below, in the course of proving assertion (3).

Proof of (3). Let N be a non-free R-module and $M=\mathrm{syz}^i(N)$. Let $0\to K\to R^b\to M\to 0$ be exact with $b=b_0(M)$ and $K\subset \mathfrak{m}R^b$. We can write $K=K'\oplus k^r$ where K' is free of direct summands isomorphic to k. Since K' is a $R/\mathfrak{m}^2$-module generated by $b_1(M)-r$ elements, we have: $l(K')\leq(b_1(M)-r)l(R/\mathfrak{m}^2)$.

From the fact that M is a syzygy module we have $\mathrm{soc}(R^{b_0})=\mathrm{soc}(K)=\mathrm{soc}(K')\oplus k^r=$ (by Lemma 3.2) $\mathfrak{m}K'\oplus k^r$. Thus $\mathfrak{m}^2R^{b_0}=\mathfrak{m}K'\oplus k^r$, and so $l(\mathfrak{m}K')=ab_0(M)-r$. Since $l(K'/\mathfrak{m}K')=b_1(M)-r$, we have $l(K')=(b_1(M)-r)+(ab_0(M)-r)$. Thus we obtain:

$$nb_{i+1}(N)=nb_1(M)\geq ab_0(M)+r(n-1)\geq ab_0(M)=ab_i(N).$$

So, if $a>n$ the sequence $(b_i(N))_{i\geq 1}$ is strictly increasing, and the sequences of Betti numbers have uniform exponential growth.

Now supposing $a=n>1$, we see that the sequence $(b_i(N))_{i\geq 1}$ is non-decreasing. If for some $j\geq 1$, $b_{j+1}(N)=b_j(N)$, then we necessarily have $\mathrm{syz}^{j+1}(N)\simeq(R/\mathfrak{m}^2)^{b_j}$ and so, the sequence $(b_i(N))_{i\geq j+1}$ is strictly increasing. If $a=n=1$, it is easily seen that the sequence $(b_i(N))_{i\geq 1}$ is stationary.

We use a different argument for $a\leq n-1$. Since $\lim_i b_i(N)=\infty$, there exists $j\geq 1$ such that $b_{j+1}(N)>b_j(N)$. By Lemma 3.5, $b_{j+2}(N)\geq nb_{j+1}(N)-ab_j(N)>b_{j+1}(N)$. Thus, the sequence $(b_i(N))_{i\geq j}$ is strictly increasing. This completes the proof of Theorem B.

3.8. Examples. (1) *Non-exceptional rings.* The first two terms of the series $P_R^k(t)$ and $(1-nt+at^2)^{-1}$ are always identical. Using an explicit form for $b_2(k)$, [6, 4.4.3], we see that the third terms are different if $\varepsilon_1\neq n(n+1)/2-a$, where ε_1 is the first deviation of the ring R.

There exists a ring isomorphism $R\simeq S/J$ where $(S,\mathfrak{n})$ is a regular local ring and J is an ideal such that $\mathfrak{n}^3\subset J\subset\mathfrak{n}^2$. It is known that $\varepsilon_1=\dim_k J/\mathfrak{n}J$, [6, 1.4.15].

We may consider a minimal system of generators of $J: x_1,\dots,x_p,\dots,x_{\varepsilon_1}$, such that $x_i\notin\mathfrak{n}^3$ if $i\leq p$ and $x_i\in\mathfrak{n}^3$ if $p<i$. Let I be the ideal generated by $x_1,\dots,x_p$. From the exact sequence

$$0 \to J/\mathfrak{n}^3 \to \mathfrak{n}^2/\mathfrak{n}^3 \to \mathfrak{n}^2/J \to 0,$$

and using the isomorphisms $J/\mathfrak{n}^3 \simeq I/\mathfrak{n}^3$, $\mathfrak{n}^2/J \simeq \mathfrak{m}^2$, we see that the preceding condition on ε_1 means that $I \neq J$, or in other words, in a minimal system of generators of J, there are elements of $\mathfrak{n}^3$.

Let $R' = S/I$ with maximal ideal $\mathfrak{m}'$. Then $R \simeq R'/\mathfrak{m}'^3$. If $I \neq J$ then $\mathfrak{m}'^3 = \mathfrak{m}' \cdot \mathfrak{m}'^2 \neq 0$, thus, Theorem A.3 applies, and by using the result of Lemma 3.5, the following improvement of Theorem B can be obtained: If $a \leq n$, the sequence $(b_i(N))_{i \geq 3}$ is strictly increasing.

(2) *Exceptional rings.* Let $R = k[X_1, \dots, X_n]/I$, where I is an ideal containing $(X_1, \dots, X_n)^3$ and generated by a set of monomials of degree two in the X_i. Then it is known that $P_R^k(t) = (1 - nt + at^2)^{-1}$, [3]. If for all i, $1 \leq i \leq n$, $X_i \cdot (X_1, \dots, X_n) \not\subset I$, then $\mathrm{soc}(R) = \mathfrak{m}^2$ and R is exceptional.

Consider in particular the ring $R = k[X_1, \dots, X_p, Y_1, \dots, Y_q]/I$ where $I = (X_1, \dots, X_p)^2 + (Y_1, \dots, Y_q)^2$ and $q \leq p$. We have $P_R^k(t) = (1-pt)^{-1}(1-qt)^{-1}$. Letting $N = R/(Y_1, \dots, Y_q)$, we have $\mathrm{syz}^1(N) \simeq N^q$ and, thus, $P_R^N(t) = (1-qt)^{-1}$. If $q = 1$ the sequence $(b_i(N))_{i \geq 0}$ is constant, and if $q < p$ we have the inequality $r_N = q^{-1} > p^{-1} = r_k$.

Let $(R, \mathfrak{m})$ be a Gorenstein ring with $\mathfrak{m}^3 = 0$ and $n > 1$. This ring is exceptional as it results from the work of Sjödin [10]. Eisenbud [2, §3] has observed that the Betti numbers $b_i(M_p)$ of $M_p = \mathrm{Hom}_R(\mathrm{syz}^p(k), R)$ are strictly decreasing for $0 \leq i < p$. In fact, $b_i(M_p) = b_{p-i-1}(k)$. Since $b_{p-1}(M_p) = 1$, the sequences of Betti numbers over R do not have uniform exponential growth.

We turn now to the case $\mathrm{soc}(R) \neq \mathfrak{m}^2$. Since $\mathfrak{m}^3 = 0$, it follows from Lemma 3.2, that this is equivalent to assuming that k is a direct summand of $\mathfrak{m}$. It is easily seen that over any local ring satisfying this last condition, k is a direct summand of any second syzygy module. From this, we deduce the following proposition.

Proposition 3.9. *Let* $(R, \mathfrak{m})$ *be a local ring with* $\mathfrak{m}^3 = 0$, *and* $\mathfrak{m}^2 \neq 0$. *Assume* $\mathrm{soc}(R) \neq \mathfrak{m}^2$. *Then:*

(1) *The sequences of Betti numbers have uniform exponential growth.*

(2) *For all non-free R-modules M, we have* $r_M = r_k$ *and the sequence* $(b_i(M))_{i \geq 1}$ *is strictly increasing.*

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